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Stochastic equations in Hilbert space with a multiplicative fractional Gaussian noise ☆

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Abstract

In this paper, some explicit solutions are given for stochastic differential equations in a Hilbert space with a multiplicative fractional Gaussian noise. This noise is the formal derivative of a fractional Brownian motion with the Hurst parameter in the interval $(1/2, 1)$. These solutions can be weak, strong or mild depending on the specific assumptions. The problem of stochastic stability of these equations is considered and for various notions of stability, sufficient conditions are given for stability. The noise may stabilize or destabilize the corresponding deterministic solutions. Various examples of stochastic partial differential equations are given that satisfy the assumptions for explicit solutions or stability.

Keywords: Fractional Brownian motion; Stochastic differential equations in a Hilbert space; Explicit solutions of linear stochastic differential equations; Fractional Gaussian noise

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1. Introduction

Fractional Brownian motion is a family of Gaussian processes that is indexed by the Hurst parameter $H \in (0,1)$. These processes with values in $\mathbb{R}^n$ were introduced by Kolmogorov [15] and some useful properties of these processes were given by Mandelbrot and Van Ness [18]. These processes seem to be applicable as models in many fields based on empirical data such as hydrology [13], economic data [17] and telecommunications [16]. Since a fractional Brownian motion for $H \neq 1/2$ is not a semimartingale, it is necessary to provide a stochastic calculus for those processes. In recent years there have been various approaches to a stochastic calculus for these processes, especially for $H \in (1/2, 1)$ (e.g., [1,5,6,25]).

In this paper, some explicit solutions are given for a family of stochastic linear equations in a Hilbert space with a finite dimensional multiplicative fractional Gaussian noise. A fractional Gaussian noise is the formal derivative of a fractional Brownian motion. For stochastic differential equations with a fractional Brownian motion, the results for existence and uniqueness are limited so it is necessary to consider solutions of special classes of stochastic differential equations. A solution for a stochastic differential equation where the diffusion coefficient is deterministic can be obtained from the solution of the corresponding deterministic equation. However, a stochastic differential equation where the diffusion term is stochastic requires a nontrivial use of stochastic calculus for a fractional Brownian motion.

For linear stochastic differential equations in a Hilbert space with multiplicative Brownian motion, Da Prato and Zabczyk [2] have given explicit solutions. The approach that is used here for linear stochastic equations with multiplicative fractional Brownian motion is motivated by Da Prato and Zabczyk [2], but the analysis for this case requires additional methods. The previous work on stochastic differential equations in a Hilbert space with a fractional Brownian motion ($H \neq 1/2$) is quite limited. For $H \in (1/2, 1)$, linear and semilinear stochastic equations with an additive fractional Brownian motion have been considered in [7,8] and a pathwise or nonprobabilistic approach has been used for some linear stochastic equations with multiplicative fractional Brownian motion in [19]. In [9–11] stochastic heat equations driven by a multiparameter fractional noise are studied. In [20] the large time behavior of random dynamical systems, that are defined by semilinear equations perturbed by a fractional noise, is investigated. Some linear evolution equations with a multiplicative fractional noise are also studied in [24] where a fractional Feynman–Kac formula is obtained. Solutions for linear stochastic equations with a multiplicative fractional Brownian motion in a finite dimensional space are given in [4].

In Section 2, an explicit solution is given to a stochastic differential equation in a Hilbert space with a multiplicative fractional Brownian motion. Depending on the specific assumptions on the linear operators in the equation, the solution can be strong, weak or mild. Some examples of stochastic partial differential equations are given that satisfy the assumptions for the solution. In Section 3, the problem of stability of these solutions is considered. The notions of stochastic stability that are considered are called pathwise exponential stability, moment stability and mean integral stability. Some examples of stochastic partial differential equations are given that satisfy the assumptions for stability. Furthermore, some examples show that the fractional noise can stabilize the system while other examples show that the fractional noise can destabilize the system.

2. A stochastic equation with multiplicative noise

A real-valued standard fractional Brownian motion with Hurst parameter $H \in (0,1)$ $(\beta^H(t), t \geqslant 0)$ is a Gaussian process with continuous sample paths such that $\mathbb{E}[\beta^H(t)] = 0$ and

$$\mathbb{E}[\beta^H(s)\beta^H(t)] = \tfrac{1}{2}[s^{2H} + t^{2H} - |t-s|^{2H}]$$

for $s, t \in \mathbb{R}_+$.

A stochastic equation in a Hilbert space with a multiplicative fractional Gaussian noise is described. Consider the stochastic equation

$$\mathrm{d}X(t) = A(t)X(t)\,\mathrm{d}t + \sum_{k=1}^{m} B_k X(t)\,\mathrm{d}\beta_k^H(t),$$

$$X(0) = x_0, \tag{2.1}$$

where $X(t)$, $x_0 \in V$, V is a separable Hilbert space, $(\beta_k^H(t), t \geqslant 0, k \in \{1, \dots, m\})$ is a family of independent real-valued standard fractional Brownian motions with the same, fixed Hurst parameter $H \in (1/2, 1)$ that are defined on a complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$, $(A(t), t \in [0,T])$ for $T > 0$ fixed and $(B_k, k \in \{1, \dots, m\})$ are typically linear densely defined operators on V. Recall that a family of bounded linear operators $(U_0(t,s), 0 \leqslant s \leqslant t \leqslant T)$ is said to be a strongly continuous evolution system corresponding to the linear operators $(A(t), t \in [0,T])$ if the map $U_0(t,s) \mapsto \mathscr{L}(V)$ is a strongly continuous function on $0 \leqslant s \leqslant t \leqslant T$, the composition property $U_0(t,r)U_0(r,s) = U_0(t,s)$ is satisfied for $0 \leqslant s \leqslant r \leqslant t \leqslant T$ and the following two equations:

$$\frac{\partial}{\partial t} U_0(t,s) = A(t)U_0(t,s) \tag{2.2}$$

and

$$\frac{\partial}{\partial s} U_0(t,s) = -U_0(t,s)A(s) \tag{2.3}$$

are satisfied on suitable domains in V (e.g., [23, Chapter 4]).

The following assumptions are used in this paper.

(A1) The family of closed operators $(A(t), t \in [0,T])$ defined on a common domain $D := \mathrm{Dom}(A(t))$ for $t \in [0,T]$ generates a strongly continuous evolution operator $(U_0(t,s), 0 \leqslant s \leqslant t \leqslant T)$ on V.

(A2) The collection of linear operators $(B_1, \ldots, B_m)$ generate mutually commuting strongly continuous groups $(S_1(s), \ldots, S_m(s), s \in \mathbb{R})$ which commute with $A(t)$ on D for each $t \in [0, T]$. For $i, j \in \{1, \ldots, m\}$, $\mathrm{Dom}(B_i B_j) \supset D$, $\mathrm{Dom}(A^*(t)) = D^*$ is independent of t and $D^* \subset \bigcap_{i,j=1}^m \mathrm{Dom}(B_i^* B_j^*)$ where $*$ denotes the topological adjoint.

(A3) The family of linear operators $(\tilde{A}(t), t \in [0, T])$ where

$$\tilde{A}(t) = A(t) - Ht^{2H-1} \sum_{j=1}^{m} B_j^2,$$

$\mathrm{Dom}(\tilde{A}(t)) = D$ for each $t \in [0, T]$, generates a strongly continuous evolution operator on V, $(U(t, s), 0 \leqslant s \leqslant t \leqslant T)$.

Initially some more specific conditions on $(A(t), t \in [0, T])$ and $(B_1, \ldots, B_m)$ are made that ensure (A1) and (A3). These conditions are particularly useful in applications to stochastic partial differential equations (SPDEs) of parabolic type that are considered subsequently.

(H1) For each $t \in [0, T]$, the linear operator $A(t)$ is a closed, densely defined operator in V whose resolvent set $\rho(A(t))$ contains the half-plane $\mathrm{Re}\, \lambda \geqslant \omega_0$ for some fixed $\omega_0 \in \mathbb{R}$ and the resolvent R satisfies the following inequality:

$$|R(\lambda, A(t))|_{\mathscr{L}(V)} \leqslant \frac{M}{1 + |\lambda + \omega_0|} \tag{2.4}$$

for $\mathrm{Re}\, \lambda \geqslant \omega_0$ and some $M > 0$ that does not depend on t.

(H2) The domain $\mathrm{Dom}(A(t)) = D$, where D has the topology from the graph norm of $A(0)$, does not depend on $t \in [0, T]$ and $-A(t)A(0)^{-1}$ is a Hölder continuous function in $\mathscr{L}(V)$, or equivalently the inequality

$$|A(t) - A(s)|_{\mathscr{L}(D,V)} \leqslant K|t - s|^{\gamma} \tag{2.5}$$

is satisfied for $s, t \in [0, T]$ are some $K > 0$ and $\gamma \in (0, 1]$.

Condition (H1) implies that $A(t)$ generates an analytic semigroup for each fixed $t \in [0, T]$. Without loss of generality, it can be assumed that $\omega_0 = 0$. Since the operator $A := -A(0)$ is strictly positive, the fractional powers A^{α} for $\alpha \in (0, 1]$ can be defined (e.g., [21]). It is well known (e.g., [23, Theorem 5.2.1]) that the hypotheses (H1) and (H2) imply (A1) and, furthermore, $\mathrm{Range}(U(t, s)) \subset D$,

$$\left| \frac{\partial}{\partial t} U_0(t, s) \right|_{\mathscr{L}(V)} = |A(t) U_0(t, s)|_{\mathscr{L}(V)} \leqslant c(t - s)^{-1} \tag{2.6}$$

for $0 \leqslant s < t \leqslant T$ and

$$|U(t, s)|_{\mathscr{L}(D)} \leqslant c \tag{2.7}$$

for some $c > 0$.

The following result shows that, with some additional conditions on the family of operators $(B_i, i = 1, 2, \ldots, m)$, (H1) and (H2), imply (A3).

Proposition 2.1. *Let* $(B_1^2, \ldots, B_m^2)$ *be closed linear operators. If* (H1) *and* (H2) *are satisfied and* $\mathrm{Dom}(B_j^2) \supset \mathrm{Dom}(A^\alpha)$ *for some* $\alpha \in [0,1)$ *and all* $j \in \{1, \ldots, m\}$ *then the family of operators* $(\tilde{A}(t), t \in [0,T])$ *where* $\mathrm{Dom}(\tilde{A}(t)) = D$ *generates a strongly continuous evolution operator on* V, *that is*, (A3) *is satisfied.*

Proof. By [21, Corollary 2.6.11], there is a constant $C>0$ such that

$$\left| Ht^{2H-1} \sum_{j=1}^{m} B_j^2 x \right| \leqslant CHT^{2H-1}(\rho^\alpha |x| + \rho^{\alpha-1}|Ax|) \tag{2.8}$$

for each $\rho > 0$ and $x \in D$. Choosing ρ sufficiently large, there is an $a \in (0, (1/2)(M+1)^{-1}H^{-1}T^{1-2H})$ such that

$$\begin{aligned}
&\left| Ht^{2H-1} \sum_{j=1}^{m} B_j^2 R(\lambda, A(t)) \right|_{\mathscr{L}(V)} \\
&\quad \leqslant Ht^{2H-1} a |A(t)R(\lambda, A(t))|_{\mathscr{L}(V)} + bHt^{2H-1}|R(\lambda, A(t))|_{\mathscr{L}(V)} \\
&\quad \leqslant Ht^{2H-1} a(M+1) + bHt^{2H-1}\frac{M}{1+|\lambda|} \\
&\quad \leqslant \frac{1}{2} + bHt^{2H-1}\frac{M}{1+|\lambda|}
\end{aligned} \tag{2.9}$$

for some constant $b>0$. Thus for λ satisfying $\mathrm{Re}\,\lambda > 2bHT^{2H-1}M - 1$ there is the inequality

$$\left| Ht^{2H-1} \sum_{j=1}^{m} B_j^2 R(\lambda, A(t)) \right|_{\mathscr{L}(V)} < c < 1$$

for each $t \in [0,T]$ which yields the inequality

$$\begin{aligned}
|R(\lambda, \tilde{A}(t))|_{\mathscr{L}(V)} &= \left| R(\lambda, A(t)) \left(I - Ht^{2H-1} \sum_{j=1}^{m} B_j^2 R(\lambda, A(t)) \right)^{-1} \right|_{\mathscr{L}(V)} \\
&\leqslant \frac{M}{1+|\lambda|}\frac{1}{1-c}
\end{aligned} \tag{2.10}$$

which verifies (2.4) with $A(t)$ replaced by $\tilde{A}(t)$. It remains to verify that the family of operators $(\tilde{A}(t), t \in [0,T])$ is Hölder continuous in $\mathscr{L}(D,V)$. By (2.5) there is the inequality

$$\begin{aligned}
&|\tilde{A}(t) - \tilde{A}(s)|_{\mathscr{L}(D,V)} \\
&\quad \leqslant |A(t) - A(s)|_{\mathscr{L}(D,V)} + H|t^{2H-1} - s^{2H-1}| \left| \sum_{j=1}^{m} B_j^2 \right|_{\mathscr{L}(D,V)} \\
&\quad \leqslant K|t-s|^\gamma + K_1|t-s|^{2H-1}
\end{aligned}$$

for $t, s \in [0, T]$ and some constant $K_1 > 0$ because $B_j^2 A^{-1} \in \mathscr{L}(V)$ by the closed graph theorem. $\square$

Some notions of solutions are given now.

Definition 2.2. A $\mathscr{B}([0,T]) \otimes \mathscr{F}$ measurable stochastic process $(X(t), t \in [0,T])$ is said to be

(i) a *strong solution* of (2.1) if $X(t) \in D$ a.s. $\mathbb{P}$ and

$$X(t) = x_0 + \int_0^t A(s)X(s)\,\mathrm{d}s + \sum_{j=1}^m \int_0^t B_j X(s)\,\mathrm{d}\beta_j^H(s) \quad \text{a.s.} \tag{2.11}$$

for $t \in [0, T]$.

(ii) a *weak solution* of (2.1) if for each $z \in D^*$

$$\begin{aligned}\langle X(t), z\rangle = \langle x_0, z\rangle &+ \int_0^t \langle X(s), A^*(s)z\rangle\,\mathrm{d}s \\ &+ \sum_{j=1}^m \int_0^t \langle X(s), B_j^* z\rangle\,\mathrm{d}\beta_j^H(s) \quad \text{a.s.}\end{aligned} \tag{2.12}$$

for $t \in [0, T]$ and

(iii) a *mild solution* of (2.1) if

$$X(t) = U_0(t,0)x_0 + \sum_{j=1}^m \int_0^t U_0(t,s)B_j X(s)\,\mathrm{d}\beta_j^H(s) \quad \text{a.s.} \tag{2.13}$$

for $t \in [0, T]$,

where all of the integrals in (2.11)–(2.13) must be well defined. For the definition of the stochastic integrals in (2.11)–(2.13) cf. [1,5,6].

Let $D_q V$ for $q \in [0, T]$ be the path or Malliavin derivative (of a smooth random variable on $(\Omega, \mathscr{F}, \mathbb{P})$) with respect to the fractional Brownian motion $(\beta_1^H(t), \dots, \beta_m^H(t), t \in [0,T])$ and define $\phi_H(r) = H(2H-1)|r|^{2H-2}$.

The main result of this section is the following theorem which gives an explicit solution to (2.1).

Theorem 2.3. *Assume that* (A1)–(A3) *are satisfied. There is a weak solution of* (2.1). *If* $x_0 \in D$, *then there is a strong solution of* (2.1). *If* $B_j \in \mathscr{L}(V)$ *for* $j \in \{1, \dots, m\}$, *then there is a mild solution of* (2.1). *In each case the solution* $(X(t), t \in [0,T])$ *is given as follows*:

$$X(t) = \prod_{j=1}^m S_j(\beta_j^H(t))U(t,0)x_0 \tag{2.14}$$

for $t \in [0, T]$.

Proof. Initially let $x_0 \in D$ and for a fixed $y \in \bigcap_{j=1}^m \mathrm{Dom}(B_j^{*2})$ define

$$v(t,x) := \left\langle U(t,0)x_0, \prod_{j=1}^m S_j^*(x_j)y \right\rangle \tag{2.15}$$

for $t \in [0,T]$ and $x = (x_1, \dots, x_m) \in \mathbb{R}^m$. Clearly $v \in C^{1,2}(\mathbb{R} \times \mathbb{R}^m)$ and the partial derivatives are:

$$D_t v(t,x) = \left\langle \tilde{A}(t)U(t,0)x_0, \prod_{j=1}^m S_j^*(x_j)y \right\rangle,$$

$$D_{x_i} v(t,x) = \left\langle U(t,0)x_0, \prod_{j=1}^m S_j^*(x_j)B_i^* y \right\rangle,$$

$$D_{x_i x_i} v(t,x) = \left\langle U(t,0)x_0, \prod_{j=1}^m S_j^*(x_j)(B_i^*)^2 y \right\rangle,$$

for $i \in \{1, \dots, m\}$.

By an Itô formula for a finite dimensional fractional Brownian motion applied to $(v(t, \beta_1^H(t), \dots, \beta_m^H(t)), t \in [0,T])$ (e.g., [6]) it follows that

$$\begin{aligned} &\mathrm{d}v\big(t, \beta_1^H(t), \dots, \beta_m^H(t)\big) &(2.16)\\ &\quad = \left\langle \tilde{A}(t)U(t,0)x_0, \prod_{j=1}^m S_j^*(\beta_j^H(t))y \right\rangle \mathrm{d}t \\ &\qquad + \sum_{j=1}^m \left\langle U(t,0)x_0, \prod_{j=1}^m S_j^*(\beta_j^H(t))(B_i^*)^2 y \right\rangle \int_0^T 1_{[0,t]}(r)\phi_H(r-t)\,\mathrm{d}r\,\mathrm{d}t \\ &\qquad + \sum_{j=1}^m \left\langle U(t,0)x_0, \prod_{j=1}^m S_j^*(\beta_j^H(t))B_i^* y \right\rangle \mathrm{d}\beta_i^H(t) &(2.17) \end{aligned}$$

for $t \in [0,T]$.

For $X(t)$ defined by equality (2.14) it follows that

$$\begin{aligned} \langle X(t), y \rangle &= v(t, \beta_1^H(t), \dots, \beta_m^H(t)) \\ &= v(0,0) \\ &\quad + \int_0^t \left\langle \left(A(s) - Hs^{2H-1} \sum_{j=1}^m B_j^2 \right) U(s,0)x_0, \prod_{j=1}^m S_j^*(\beta_j^H(s))y \right\rangle \mathrm{d}s \\ &\quad + \sum_{i=1}^m \int_0^t Hs^{2H-1} \left\langle B_i^2 U(s,0), \prod_{j=1}^m S_j^*(\beta_j^H(s))y \right\rangle \mathrm{d}s \\ &\quad + \sum_{j=1}^m \int_0^t \left\langle U(s,0)x_0, \prod_{j=1}^m S_j^*(\beta_j^H(s))B_i^* y \right\rangle \mathrm{d}\beta_i^H(s) \quad \text{a.s.} \end{aligned} \tag{2.18}$$

for $t \in [0, T]$ and therefore

$$\begin{aligned}\langle X(t), y\rangle &= \langle x_0, y\rangle + \int_0^t \left\langle A(s)\prod_{j=1}^m S_j(\beta_j^H(s))U(s,0)x_0, y\right\rangle \mathrm{d}s \\ &\quad + \sum_{i=1}^m \int_0^t \left\langle B_i \prod_{j=1}^m S_j\left(\beta_j^H(s)\right) U(s,0)x_0, y\right\rangle \mathrm{d}\beta_i^H(s) \\ &= \langle x_0, y\rangle + \int_0^t \langle A(s)X(s), y\rangle\,\mathrm{d}s + \sum_{j=1}^m \int_0^t \langle B_i X(s), y\rangle\,\mathrm{d}\beta_i^H(s) \quad \text{a.s.} \quad (2.19)\end{aligned}$$

for $t \in [0, T]$.

Using a countable dense family of elements $y \in \bigcap_{j=1}^m \mathrm{Dom}((B_j^*)^2)$ in (2.14), the equality (2.11) is verified, so that $X(t)$ given by (2.14) is a strong solution of (2.1).

Now let $x_0 \in V$. It is shown that $X(t)$ given by (2.14) defines a weak solution of (2.1). Let $(x_n, n \in \mathbb{N})$ be a D-valued sequence that converges to $x \in V$. Define $X_n(t) = \prod_{j=1}^m S_j(\beta_j^H(t))U(t,0)x_n$. By the preceding part of the proof, $(X_n(t), t \in [0,T])$ is a strong solution of (2.1) with the initial condition $X_n(0) = x_n$. For each $z \in D^*$

$$\langle X_n(t), z\rangle = \langle x_n, z\rangle + \int_0^t \langle X_n(s), A^*(s)z\rangle\,\mathrm{d}s + \sum_{j=1}^m \int_0^t \langle X_n(s), B_j^* z\rangle\,\mathrm{d}\beta_j^H(s) \qquad (2.20)$$

Clearly the sequence of solutions $(X_n(t), t \in [0,T], n \in \mathbb{N})$ converges uniformly a.s. to $(X(t), t \in [0,T])$ by the description of the solution (2.14), and the sequences $(\langle X_n(t), z\rangle, n \in \mathbb{N})$, $(\langle x_n, z\rangle, n \in \mathbb{N})$ and $(\int_0^t \langle X_n(s), A^*(s)z\rangle\,\mathrm{d}s, n \in \mathbb{N})$ converge to $\langle X(t), z\rangle$, $\langle x, z\rangle$ and $\int_0^t \langle X(s), A^*(s)z\rangle\,\mathrm{d}s$ a.s. respectively for each $t \in [0,T]$. To verify the convergence of the sequence of stochastic integrals obtained from (2.20) to $\sum_{j=1}^m \int_0^t \langle X(s), B_j^* z\rangle\,\mathrm{d}\beta_j^H(s)$, consider an arbitrary and fixed $j \in \{1, \ldots, m\}$ and let $\varphi_n(s) = \langle X_n(s), B_j^* z\rangle$ and $\varphi_0(s) = \langle X(s), B_j^* z\rangle$. The path derivative $D_r\varphi_n(s)$ is an m dimensional vector such that

$$(D_r\varphi_n(s))_l = \left\langle \prod_{i=1}^m S_i(\beta_i^H(s))U(s,0)x_n, B_l^* B_j^* z\right\rangle 1_{[0,s]}(r) \qquad (2.21)$$

for $r \in [0,T]$, $l \in \{1, \ldots, m\}$ and $n \in \mathbb{N}$. Since $(S_i(t), i = 1, \ldots, m, t \in \mathbb{R})$ is a finite collection of strongly continuous groups, there are positive real numbers M and ω such that

$$\sup_{s \in [0,T]} \mathbb{E}|S_i(\beta_i(s))|_{\mathscr{L}(V)} \leqslant \sup_{s \in [0,T]} M\mathbb{E}(\exp[\omega|\beta_i(s)|]) < \infty$$

for each $i \in \{1, \ldots, m\}$. Thus,

$$\begin{aligned}&\mathbb{E}\int_0^t \int_0^t \varphi_0(s)\varphi_0(r)\phi_H(r-s)\,\mathrm{d}r\,\mathrm{d}s \\ &\quad + \mathbb{E}\int_0^t \int_0^t \int_0^t \int_0^t D_p\varphi_0(q)D_r\varphi_0(s)\phi_H(p-s)\phi_H(r-q)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r\,\mathrm{d}s\end{aligned}$$

$$\leqslant \sup_{\substack{s\in[0,T]\\ j,l\in\{1,\dots,m\}}} m\mathbb{E}\prod_{i=1}^{m}|S_i(\beta_i^H(s))|^2_{\mathscr{L}(V)}|U(s,0)x_0|^2(|B_j^*z|^2+|B_l^*B_j^*z|^2)$$
$$\times\left(\int_0^t\int_0^t\phi_H(s-r)\,\mathrm{d}r\,\mathrm{d}s\right.$$
$$\left.+\int_0^t\int_0^t\int_0^s\int_0^q\phi_H(p-s)\phi_H(r-q)\,\mathrm{d}p\,\mathrm{d}r\,\mathrm{d}q\,\mathrm{d}s\right)<\infty. \tag{2.22}$$

This inequality implies that φ_0 is integrable with respect to β^H (e.g., Theorem 2 in [6]). Furthermore, there is the inequality

$$\mathbb{E}\left[\int_0^t(\varphi_n(s)-\varphi_0(s))\,\mathrm{d}\beta_j^H(s)\right]^2$$
$$=\mathbb{E}\int_0^t\int_0^t(\varphi_n(s)-\varphi_0(s))(\varphi_n(r)-\varphi_0(r))\phi_H(r-s)\,\mathrm{d}r\,\mathrm{d}s$$
$$+\mathbb{E}\int_0^t\int_0^t\int_0^t\int_0^t D_p(\varphi_n(q)-\varphi_0(q))D_r(\varphi_n(s)-\varphi_0(s))$$
$$\times\phi_H(p-s)\phi_H(r-q)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r\,\mathrm{d}s$$
$$\leqslant \sup_{\substack{s\in[0,T]\\ j,l\in\{1,\dots,m\}}} m\mathbb{E}\left[\prod_{i=1}^{m}|S_i(\beta_i^H(s))|^2(|B_j^*z|^2+|B_l^*B_j^*z|^2)\right.$$
$$\times\left(\int_0^t\int_0^t|U(s,0)(x_n-x_0)||U(r,0)(x_n-x_0)|\phi_H(s-r)\,\mathrm{d}r\,\mathrm{d}s\right.$$
$$+\int_0^t\int_0^t\int_0^s\int_0^q|U(q,0)(x_n-x_0)||U(s,0)(x_n-x_0)|$$
$$\left.\times\phi_H(p-s)\phi_H(r-q)\,\mathrm{d}p\,\mathrm{d}r\,\mathrm{d}q\,\mathrm{d}s\right). \tag{2.23}$$

The right hand side of this inequality tends to zero as $n\to\infty$ by the boundedness of $|U(t,s)|_{\mathscr{L}(V)}$ for $0\leqslant s\leqslant t\leqslant T$. Thus, there is the equality

$$\langle X(t),z\rangle=\langle x_0,z\rangle+\int_0^t\langle X(s),A^*(s)z\rangle\,\mathrm{d}s+\sum_{j=1}^m\int_0^t\langle X(t),B_j^*z\rangle\,\mathrm{d}\beta_j^H(s)\quad\text{a.s.} \tag{2.24}$$

It remains to prove that if $B_j\in\mathscr{L}(V)$ for all $j\in\{1,\dots,m\}$, then $X(t)$ given by (2.14) is a mild solution of (2.1). Initially let $x_0\in D$. For a fixed $t\in(0,T)$, $y\in V$, the real-valued function

$$v(s,x)=\left\langle U_0(t,s)\prod_{j=1}^m S_j(x_j)U(s,0)x_0,y\right\rangle$$

for $s \in [0, t)$, $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$ satisfies

$$\begin{aligned}
\frac{\partial v}{\partial s}(s,x) = & -\left\langle U_0(t,s)A(s)\prod_{j=1}^{m} S_j(x_j)U(s,0)x_0, y\right\rangle \\
& + \left\langle \prod_{j=1}^{m} S_j(x_j)\tilde{A}(s)U(s,0)x_0, U_0^*(t,s)y\right\rangle \\
= & -\left\langle A(s)\prod_{j=1}^{m} S_j(x_j)U(s,0)x_0, U_0^*(t,s)y\right\rangle \\
& + \left\langle \left(A(s) - Hs^{2H-1}\sum_{j=1}^{m} B_j^2\right)\prod_{j=1}^{m} S_j(x_j)U(s,0)x_0, U_0^*(t,s)y\right\rangle \\
= & -Hs^{2H-1}\sum_{j=1}^{m}\left\langle B_j^2\prod_{j=1}^{m} S_j(x_j)U(s,0)x_0, U_0^*(t,s)y\right\rangle, \\
\frac{\partial v}{\partial x_i}(s,x) = & \left\langle B_i\prod_{j=1}^{m} S_j(x_j)U(s,0)x_0, U_0^*(t,s)y\right\rangle, \\
\frac{\partial^2 v}{\partial x_i^2}(s,x) = & \left\langle B_i^2\prod_{j=1}^{m} S_j(x_j)U(s,0)x_0, U_0^*(t,s)y\right\rangle,
\end{aligned}$$

for $i \in \{1, \ldots, m\}$.

Apply an Itô formula (e.g., [7]) to the process $(v(s, \beta_1^H(s), \ldots, \beta_m^H(s)), s \in (0,t))$ to obtain

$$\begin{aligned}
&\left\langle \prod_{j=1}^{m} S_j(\beta_j^H(t))U(t,0)x_0, y\right\rangle \\
&\quad = \langle U_0(t,s)x_0, y\rangle \\
&\qquad + \int_0^t \left[\frac{\partial v}{\partial s}(s, \beta_1^H(s), \ldots, \beta_m^H(s)) + \sum_{j=1}^{m} Hs^{2H-1}\frac{\partial^2 v}{\partial x_j^2}(s, \beta_1^H(s), \ldots, \beta_m^H(s))\right] \mathrm{d}s \\
&\qquad + \sum_{j=1}^{m}\int_0^t \frac{\partial v}{\partial x_j}(s, \beta_1^H(s), \ldots, \beta_m^H(s))\,\mathrm{d}\beta_j^H(s) \\
&\quad = \langle U_0(t,s)x_0, y\rangle \\
&\qquad + \sum_{i=1}^{m}\int_0^t \left\langle B_i\prod_{j=1}^{m} S_j(\beta_j^H(s))U(s,0)x_0, U_0^*(t,s)y\right\rangle \mathrm{d}\beta_i^H(s).
\end{aligned} \tag{2.25}$$

Thus,

$$\begin{aligned}
\langle X(t), y\rangle = & \langle U_0(t,s)x_0, y\rangle \\
& + \sum_{i=1}^{m}\int_0^t \left\langle U_0(t,s)B_i\prod_{j=1}^{m} S_j(\beta_j^H(s))U(s,0)x_0, y\right\rangle \mathrm{d}\beta_i^H(s) \quad \text{a.s.}
\end{aligned} \tag{2.26}$$

so $(X(t), t \in [0,T])$ is a mild solution of (2.1). If $x_0 \in V$, then there is a D-valued sequence $(x_n, n \in \mathbb{N})$ that converges to x_0 and an associated sequence of processes $(X_n(t), t \in [0,T], n \in \mathbb{N})$ that converges uniformly a.s. to $(X(t), t \in [0,T])$. For $(X_n(t), t \in [0,T])$ there is the equality

$$\begin{aligned} X_n(t) &= U_0(t,0)x_n + \sum_{i=1}^{m} \int_0^t U_0(t,s)B_i X_n(s)\,\mathrm{d}\beta_i^H(s) \\ &=: U_0(t,0)x_n + \sum_{i=1}^{m} \int_0^t \psi_n^i(s)\,\mathrm{d}\beta_i^H(s). \end{aligned} \tag{2.27}$$

To show that $(X_n(t), t \in [0,T])$ is a mild solution of (2.1) with $X(0) = x_0$, it suffices to show for fixed t that

$$\lim_{n\to\infty} \int_0^t \langle \psi_n^i(s), y\rangle\,\mathrm{d}\beta_i^H(s) = \int_0^t \langle \psi_0^i(s), y\rangle\,\mathrm{d}\beta_i^H(s) \quad \text{a.s.} \tag{2.28}$$

for each $y \in V$ and $i \in \{1,\ldots,m\}$. Fix $i \in \{1,\ldots,m\}$ and let $\psi_0 = \psi_0^i$. To ensure that the stochastic integral $\int_0^t \psi_0(s)\,\mathrm{d}\beta_i^H(s)$ is well defined the following computation is made.

$$\begin{aligned} &\mathbb{E}\int_0^t \int_0^t \langle \psi_0(s), \psi_0(r)\rangle \phi_H(r-s)\,\mathrm{d}r\,\mathrm{d}s \\ &\quad + \mathbb{E}\int_0^t\int_0^t\int_0^t\int_0^t \langle D_p\psi_0(q), D_r\psi_0(s)\rangle_{\mathscr{L}(\mathbb{R}^m,V)} \\ &\quad \times \phi_H(p-s)\phi_H(r-q)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r\,\mathrm{d}s \\ &= \mathbb{E}\int_0^t\int_0^t \Bigg\langle U_0(t,s)B_i \prod_{j=1}^{m} S_j(\beta_j^H(s))U(s,0)x_0, \\ &\qquad\qquad U_0(t,r)B_i \prod_{j=1}^{m} S_j(\beta_j^H(r))U(r,0)x_0 \Bigg\rangle \phi_H(r-s)\,\mathrm{d}r\,\mathrm{d}s \\ &\quad + \mathbb{E}\int_0^t\int_0^t\int_0^t\int_0^t \sum_{l=1}^{m} \Bigg\langle U_0(t,q)B_iB_l \prod_{j=1}^{m} S_j(\beta_j^H(s))U(q,0)x_0, \\ &\qquad\qquad U_0(t,s)B_iB_l \prod_{j=1}^{m} S_j(\beta_j^H(r))U(s,0)x_0 \Bigg\rangle \\ &\quad \times 1_{[0,q]}(p)1_{[0,s]}(r)\phi_H(p-s)\phi_H(r-q)\,\mathrm{d}p\,\mathrm{d}q\,\mathrm{d}r\,\mathrm{d}s \\ &\leqslant \sup_{s\in[0,t]} |x_0|^2 |U_0(t,s)|^2_{\mathscr{L}(V)} |B_i|^2_{\mathscr{L}(V)} \mathbb{E}\prod_{j=1}^{m} |S_j(\beta_j^H(s))|^2_{\mathscr{L}(V)} \\ &\quad \times \int_0^t\int_0^t \phi_H(r-q)\,\mathrm{d}r\,\mathrm{d}q \\ &\quad + k \sup_{\substack{s\in[0,t] \\ l\in\{1,\ldots,m\}}} |U_0(t,s)|^2_{\mathscr{L}(V)} |B_i|^2_{\mathscr{L}(V)} |B_l|^2_{\mathscr{L}(V)} |x_0|^2 \mathbb{E}\prod_{j=1}^{m} |S_j(\beta_j^H(s))|^2_{\mathscr{L}(V)} \end{aligned}$$

$$\times \int_0^t \int_0^t \int_0^u \int_0^q \phi_H(p-u)\phi_H(r-q)\,\mathrm{d}p\,\mathrm{d}r\,\mathrm{d}u\,\mathrm{d}q$$
$$< \infty. \tag{2.29}$$

Thus, the process $(U_0(t,s)B_iX(0), s \in [0,t])$ is integrable with respect to β^H. The limit (2.29) can be verified as in the analogous case in (2.23). Letting $n \to \infty$ in (2.28) shows that $(X(t), t \in [0,T])$ is a mild solution of (2.1). $\square$

Some specific examples of stochastic partial differential (SPDEs) are given for which explicit solutions can be described using Theorem 2.3.

Example 2.4. Consider the following stochastic parabolic equation of $2k$th order:

$$\frac{\partial u}{\partial t}(t,\xi) = L(t,\xi)u(t,\xi) + bu\frac{\mathrm{d}\beta^H}{\mathrm{d}t},$$

$$u(0,\xi) = x_0(\xi), \tag{2.30}$$

for $(t,\xi) \in [0,T] \times \mathcal{O}$

$$\left(\frac{\partial u}{\partial \xi}\right)^{\alpha}(t,\xi) = 0, \quad (t,\xi) \in [0,T] \times \partial\mathcal{O},\ \alpha \in \{1,\dots,k-1\},$$

where $k \in \mathbb{N}$, $\mathcal{O} \subset \mathbb{R}^d$ is a bounded domain of class C^k, $b \in \mathbb{R}\backslash\{0\}$ and

$$L(t,\xi) := \sum_{|\alpha| \leqslant 2k} a_\alpha(t,\xi)D^\alpha \tag{2.31}$$

is a strongly elliptic operator on $\mathcal{O}$, uniformly in $(t,\xi) \in [0,T] \times \bar{\mathcal{O}}$ and $a_\alpha(t,\cdot) \in C^{2k}(\bar{\mathcal{O}})$ for each $t \in [0,T]$. Eq. (2.30) is rewritten in the form

$$\mathrm{d}X(t) = A(t)X(t)\,\mathrm{d}t + BX(t)\,\mathrm{d}\beta^H(t),$$

$$X(0) = x_0 \in V, \tag{2.32}$$

for $t \in [0,T]$, where $V = L^2(\mathcal{O})$, $(A(t)u)(\xi) = L(t,\xi)u(t,\xi)$, $\mathrm{Dom}(A(t)) = D = H^{2k}(\mathcal{O}) \cap H_0^k(\mathcal{O})$ and $B = bI \in \mathscr{L}(V)$. It is assumed that

$$\sup_{\xi \in \mathcal{O}} |a_\alpha(t,\xi) - a_\alpha(s,\xi)| \leqslant M|t-s|^\gamma \tag{2.33}$$

for $|\alpha| \leqslant 2k$, $s,t \in [0,t]$ and a constant M. Hypotheses (H1) and (H2) are satisfied (cf. [23, Theorem 3.8.3]), so by Proposition 2.1 the assumptions (A1) and (A3) are satisfied too. The assumption (A2) is trivially satisfied. Note that $D^* = \mathrm{Dom}(A^*(t)) = \mathrm{Dom}(A(t)) = D$. By Theorem 2.3 there is a solution of (2.32) in both the weak and the mild sense. If $x_0 \in D$, then there is a strong solution.

A second example is given.

Example 2.5. Consider the stochastic Cauchy problem

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{d} a_{ij}(t) \frac{\partial^2 u}{\partial \xi_i \partial \xi_j}(t,\xi) + \sum_{i=1}^{d} d_i(t) \frac{\partial u}{\partial \xi_i}(t,\xi) + c(t)u(t,\xi)$$
$$+ \sum_{i=1}^{d} b_i \frac{\partial u}{\partial \xi_i}(t,\xi) \frac{\mathrm{d}\beta_1^H(t)}{\mathrm{d}t} + \tilde{b}u(t,\xi) \frac{\mathrm{d}\beta_2^H(t)}{\mathrm{d}t} \tag{2.34}$$

for $(t,\xi) \in [0,T] \times \mathbb{R}^d$ and

$$u(0,\xi) = x_0(\xi),$$

where $a_{ij},\ d_i,\ c,\ i,j \in \{1,\dots,d\}$ are Hölder continuous functions and $b_i, \tilde{b} \in \mathbb{R}$, $i \in \{1,\dots,d\}$. It is assumed that the differential operator

$$L(t) := \sum_{i,j}^{d} a_{ij}(t) \frac{\partial^2}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{d} d_i(t) \frac{\partial}{\partial \xi_i} + c(t)I$$

is uniformly elliptic, that is,

$$\sum_{i,j=1}^{d} a_{ij}(t) v_i v_j > 0 \tag{2.35}$$

is satisfied for $v \in \mathbb{R}^d \backslash \{0\}$ and $t \in [0,T]$. Eq. (2.34) is rewritten as

$$\mathrm{d}X(t) = A(t)X(t)\,\mathrm{d}t + B_1 X(t)\,\mathrm{d}\beta_1^H(t) + B_2 X(t)\,\mathrm{d}\beta_2^H(t),$$

$$X(0) = x_0, \tag{2.36}$$

where $X(t), x_0 \in V$, $V = L^2(\mathbb{R}^d)$, $A(t) = L(t)$ with $\mathrm{Dom}(A(t)) = \mathrm{Dom}(A^*(t)) = H^2(\mathbb{R}^d)$, $B_1 = \sum_{i=1}^d b_i(\partial/\partial\xi_i)$, $\mathrm{Dom}(B_1) = H^1(\mathbb{R}^d)$ and $B_2 = \tilde{b}I$. It is well known that the family of operators $(A(t), t \in [0,T])$ generates a strongly continuous evolution system $(U_0(t,s), 0 \leqslant s \leqslant t \leqslant T)$ on V and (A1) is satisfied (e.g., Theorem 5.2.1 in [23]). The operators B_1 and B_2 generate strongly continuous groups on V that are given as follows:

$$[S_1(t)x_0](\xi) = x_0(\xi_1 + b_1 t, \dots, \xi_d + b_d t) \tag{2.37}$$

for $\xi = (\xi_i, \dots, \xi_d) \in \mathbb{R}^d$ and $t \in \mathbb{R}$ and

$$S_2(t)x_0 = (\mathrm{e}^{\tilde{b}t} I)x_0 \tag{2.38}$$

for $t \in \mathbb{R}$, respectively, so (A2) is clearly satisfied. To verify (A3), note that $\tilde{A}(t)$ with $\mathrm{Dom}(\tilde{A}(t)) = D = H^2(\mathbb{R}^d)$ is also a second order differential operator with time dependent Hölder continuous coefficients. It is only necessary to ensure that the operator is uniformly elliptic on $[0,T]$. The highest order term is

$$L_0(t) = \sum_{i,j=1}^{d} (a_{ij}(t) - Ht^{2H-1} b_i b_j) \frac{\partial^2}{\partial \xi_i \partial \xi_j}, \tag{2.39}$$

so the ellipticity condition is

$$\sum_{i,j=1}^{d} a_{ij}(t)v_iv_j > Ht^{2H-1}\sum_{i,j=1}^{d} b_ib_jv_iv_j \tag{2.40}$$

for $t \in [0,T]$ and $v \in \mathbb{R}^d\backslash\{0\}$. If (2.40) is satisfied, then Theorem 2.3 can be applied to obtain a strong solution for $x_0 \in D = H^2(\mathbb{R}^d)$ or a weak solution of (2.36) defined by (2.34). If $(a_{ij}(t)) = (a_{ij})$ is a constant positive definite matrix, then condition (2.40) is always satisfied for sufficiently small intervals $[0,T]$ but for t sufficiently large inequality (2.40) can be violated. Thus, there is a solution on time intervals $[0,T]$ with $T>0$ but bounded to ensure the strong ellipticity condition (2.40).

To demonstrate more explicitly the phenomenon associated with (2.36), consider the special case of the one dimensional equation

$$\frac{\partial u}{\partial t}(t,\xi) = a\,\frac{\partial^2 u}{\partial \xi^2}(t,\xi) + b\,\frac{\partial u}{\partial \xi}(t,\xi)\frac{\mathrm{d}\beta^H}{\mathrm{d}t}(t) \tag{2.41}$$

and

$$u(0,\xi) = x_0(\xi)$$

for $t>0$ and $\xi \in \mathbb{R}$ where $a>0$ and $b \in \mathbb{R}\backslash\{0\}$. The ellipticity condition (2.40) is

$$a > Ht^{2H-1}b^2. \tag{2.42}$$

From the preceding analysis, the solution is defined on intervals $[0,T]$ where $T \leqslant T_1 = (a/b^2H)^{1/(2H-1)}$. In this case, more information can be obtained. The solution is given as

$$X(t) = S_1(\beta^H(t))U(t,0)x_0,$$

where $[S_1(s)x](\xi) = x(\xi + bs)$ and U is the evolution operator corresponding to the equation

$$\frac{\partial y}{\partial t} = (a - Ht^{2H-1}b^2)\frac{\partial^2 y}{\partial \xi^2},$$

$$y(0) = x_0.$$

Thus, U may be determined by a time composition. If S_Δ is the heat semigroup on $\mathbb{R}$

$$(S_\Delta x)(\xi) = \int_{\mathbb{R}} (4\pi t)^{-1/2}\exp\left[-\frac{1}{4t}(\xi-\eta)^2\right]x(\eta)\,\mathrm{d}\eta \tag{2.43}$$

then

$$X(t) = S_1(\beta^H(t))S_\Delta\left(at - \frac{1}{2}b^2t^{2H}\right)x_0. \tag{2.44}$$

Thus $X(t)$ is well defined if $at-(1/2)b^2t^{2H}\geqslant 0$, that is, for $t\in[0,T_2]$ where

$$T_2=\left(\frac{2a}{b^2}\right)^{1/(2H-1)}.$$

In fact, the transformed time in the semigroup S_Δ initially increases from zero, but at the time T_1 it begins to decrease so that at T_2 it returns to zero. In general, the solution cannot be extended beyond T_2 because the heat semigroup S_Δ is not defined for negative times. It may leave the space $V=L^2(\mathbb{R}^d)$ after T_2. However, for a suitably chosen initial condition x_0 it can be continued where the problem corresponds to the ill-posed parabolic problem with reversed time.

3. Stability of solutions

In this section it is assumed that (A1)–(A3) are satisfied for each $T\in(0,\infty)$, so that the suitable various solutions of (2.1) exist on $\mathbb{R}_+$ by Theorem 2.3. The limit behavior of the solutions as $t\to\infty$ is investigated.

Eq. (2.1) is said to be *pathwise exponentially stable* if the inequality

$$|X(t)|\leqslant M\mathrm{e}^{-\omega t}|x_0|\quad \text{a.s. } \mathbb{P} \tag{3.1}$$

for $X(0)=x_0\in V$, each $t\in\mathbb{R}_+$, some $\omega>0$ and a positive random variable M, where $(X(t),t\in\mathbb{R}_+)$ is the solution of (2.1) given by (2.13).

Eq. (2.1) is said to be *exponentially stable in the* $2p$th *moment* (for $p=1$ it is usually called mean square exponentially stable) for a fixed $p>0$ if the inequality

$$\mathbb{E}|X(t)|^{2p}\leqslant C\mathrm{e}^{-\omega t}|x_0|^{2p} \tag{3.2}$$

is satisfied for some positive constant C and some $\omega>0$, where $(X(t),t\in\mathbb{R}_+)$ is the solution of (2.1) given by (2.14).

Eq. (2.1) is said to be $2p$th *stable in the mean* if

$$\mathbb{E}\int_0^\infty |X(t)|^{2p}\,\mathrm{d}t\leqslant C|x_0|^{2p} \tag{3.3}$$

is satisfied for some positive constant C, where $(X(t),t\in\mathbb{R}_+)$ is the solution of (2.1) given by (2.14).

For the Wiener process ($H=1/2$) it is known that (3.2) and (3.3) are equivalent [14], which is an analogue of a well known result of Datko [3] for semigroups. However, to prove this equivalence the Markov property is used. A solution of (2.1) with $H>1/2$ is not Markov. Clearly (3.2) implies (3.3).

For simplicity of presentation it is assumed that there is only one scalar fractional Brownian motion in (2.1), that is, $m=1$, $\beta_1^H=\beta^H$, $B_1=B$ because the general case with $m>1$ is analogous.

The following result provides conditions for pathwise exponential stability.

Proposition 3.1. *Let* (A1)–(A3) *and*

$$\langle A(t)x - Ht^{2H-1}B^2x, x\rangle \leqslant -\tilde{\omega}(t)|x|^2 \tag{3.4}$$

for $x \in D$, *where* $\tilde{\omega} : \mathbb{R}_+ \to \mathbb{R}$ *is continuous, be satisfied. Then the following inequality is satisfied*:

$$|X(t)| \leqslant M_1 \exp\left[k_1 t^H \sqrt{\log\log t} - \int_0^t \tilde{\omega}(s)\,\mathrm{d}s\right]|x_0| \tag{3.5}$$

for $t > e$ *and* $x_0 \in V$ *where* M_1 *is a positive random variable and* $k_1 > 0$. *If* $\tilde{\omega}(s) \geqslant \omega_0 > 0$ *for all* $s \in \mathbb{R}_+$, *then Eq.* (2.1) *is pathwise exponentially stable.*

Proof. If $x_0 \in D$, then $X(t) = S(\beta^H(t))U(t,0)x_0$ and the function $y(t) = U(t,0)x_0$ satisfies the equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \tilde{A}(t)y = (A(t) - Ht^{2H-1}B^2)y$$

$$y(0) = x_0. \tag{3.6}$$

Thus

$$\begin{aligned}\frac{\mathrm{d}|y(t)|^2}{\mathrm{d}t} &= 2\langle (A(t) - Ht^{2H-1}B^2)y(t), y(t)\rangle \\ &\leqslant -2\tilde{\omega}(t)|y(t)|^2\end{aligned}$$

for $t \geqslant 0$ so integrating this inequality yields

$$|y(t)|^2 \leqslant |x_0|^2 \mathrm{e}^{-2\int_0^t \tilde{\omega}(s)\,\mathrm{d}s} \tag{3.7}$$

for $x_0 \in D$ and $t \geqslant 0$. Since S is a strongly continuous group, it follows that

$$|S(\beta^H(t))|_{\mathscr{L}(V)} \leqslant k_1 \mathrm{e}^{k_2|\beta^H(t)|} \tag{3.8}$$

for $t \geqslant 0$ and some positive real numbers k_1 and k_2. By the Law of the Iterated Logarithm for a fractional Brownian motion [12] there is the equality

$$\limsup_{t\to\infty} \frac{|\beta^H(t)|}{t^H\sqrt{\log\log t}} = c_H \quad \text{a.s.} \tag{3.9}$$

where c_H is a real number that only depends on H. Given $\varepsilon > 0$ there is a random time $\bar{T}$ such that if $\tau \geqslant \bar{T}$ then $|\beta^H(\tau)|/\tau^H\sqrt{\log\log \tau} \leqslant c_H + \varepsilon$ a.s. Thus for $t \geqslant 0$ there is a random variable M_1 that depends on $(\beta^H(\tau), \tau \in [0, \bar{T}])$ such that

$$\begin{aligned}|X(t)| &\leqslant |S(\beta^H(t))|_{\mathscr{L}(V)}|y(t)| \\ &\leqslant M_1 \exp\left[(c_H + \varepsilon)t^H\sqrt{\log\log t} - \int_0^t \tilde{\omega}(s)\,\mathrm{d}s\right]|x_0|\end{aligned} \tag{3.10}$$

for $x_0 \in D$. If $x_0 \in V$, then there is a D-valued sequence $(x_n, n \in \mathbb{N})$ that converges to x_0 in V. The corresponding sequence of solutions of (2.1) $(X_n(t), t \geqslant 0)$ converges to $X(t)$ a.s. for each $t \in \mathbb{R}_+$ so the inequality (3.5) is satisfied. $\square$

The above approach can also be used to obtain some results on exponential stability in the $2p$th moment. To obtain more precise results the following result based on an Itô formula for fractional Brownian motion is useful. Recall that the singleton set $\{0\}$ is said to be nonattainable if it is never hit (a.s.) by $(X(t), t \geqslant 0)$ in a finite time for $x_0 \neq 0$. Clearly, if $S(s)x_0 \neq 0$ and $U(t,0)x_0 \neq 0$ for each $x_0 \in V \backslash \{0\}$, $s \in \mathbb{R}$, $t \in \mathbb{R}_+$, then $\{0\}$ is nonattainable. This is the case in both examples in the previous section.

Theorem 3.2. *If* (H1), (H2) *and* (A2) *are satisfied and the following Lyapunov type inequality is satisfied*

$$\langle A(t)x, x\rangle |x|^2 + 2(p-1)Ht^{2H-1}\langle x, Bx\rangle^2 + Ht^{2H-1}|Bx|^2|x|^2 \leqslant -\omega(t)|x|^4 \quad (3.11)$$

for $x \in D$, $t \in \mathbb{R}_+$ *where* $\omega : \mathbb{R}_+ \to \mathbb{R}$ *is continuous, then for* $p \geqslant 1$

$$\mathbb{E}|X(t)|^{2p} \leqslant |x_0|^{2p} \exp\left[-2p \int_0^t \omega(s)\, \mathrm{d}s\right] \quad (3.12)$$

for $t \geqslant 0$ *and* $x_0 \in V$. *If* $\omega(t) \geqslant \omega_0 > 0$ *for all* $t \in \mathbb{R}_+$ *then Eq.* (2.1) *is exponentially stable in the* $2p$th *moment. Furthermore if* $\{0\}$ *is nonattainable, then the former conclusions are satisfied for each* $p > 0$.

Remark 3.3. Before giving a proof of Theorem 3.2 some interesting special cases of the Lyapunov inequality (3.11) are noted.

(i) For the mean square exponential stability ($p = 1$), the inequality (3.11) reduces to

$$\langle A(t)x, x\rangle + Ht^{2H-1}|Bx|^2 \leqslant -\omega(t)|x|^2 \quad (3.13)$$

for $x \in D$ and $t \in \mathbb{R}_+$.

(ii) If $B \in \mathscr{L}(V)$ and $\langle A(t)x, x\rangle \leqslant -\alpha(t)|x|^2$ for $x \in D$ and $t \in \mathbb{R}_+$ then (3.11) is satisfied with $\omega(t) = \alpha(t) - Ht^{2H-1}|B|^2_{\mathscr{L}(V)}(2|p-1|+1)$.

(iii) If $B = bI$ then inequality (3.11) is

$$\langle A(t)x, x\rangle + (2p-1)b^2Ht^{2H-1}|x|^2 \leqslant -\omega(t)|x|^2 \quad (3.14)$$

for $x \in D$ and $t \in \mathbb{R}_+$.

Note that for $p < 1/2$, the fractional Gaussian noise has a stabilizing effect.

Proof of Theorem 3.2. Initially assume that $\{0\}$ is nonattainable, $p > 0$ and $x_0 \in D$. Let $V(x) = |x|^{2p}$ and note that the first two derivatives are $D_x V(x) = 2p|x|^{2(p-1)}x$ and $D^2_{xx}V(x) = 4p(p-1)|x|^{2(p-2)}x \circ x + 2p|x|^{2(p-1)}I$. Since V is a Hilbert space, the first derivative $D_x V(x)$ is identified with an element of V and the second derivative is identified with a symmetric element of $\mathscr{L}(V)$.

For $R>1$ let V_R be a $C^2(V\backslash\{0\})$ function such that $V_R(x)1_{\{|x|\leqslant R\}} = V(x)1_{\{|x|\leqslant R\}}$, the three functions V_R, $D_x V_R$ and $D^2_{xx} V_R$ are bounded on $V\backslash\{|x|\leqslant R\}$ and $0\leqslant V_R(x)\leqslant V(x)$, $|D_x V_R(x)|\leqslant|D_x V(x)|$, $|D^2_{xx} V_R(x)|_{\mathscr{L}(V)}\leqslant|D^2_{xx} V(x)|_{\mathscr{L}(V)}$ for $x\in V\backslash\{|x|\leqslant R|\}$. Fix $R>1$. For a $\delta\in(0,1)$ choose a $V_{R,\delta}$ in $C^2(V)$ so that $V_{R,\delta}(x) = V_R$ for $|x|\geqslant\delta$, the following inequalities are satisfied for $V_{R,\delta}$, $0\leqslant V_{R,\delta}(x)\leqslant V_R(x){+}1$, $|D_x V_{R,\delta}(x)|\leqslant|D_x V_R(x)|$, $|D^2_{xx} V_{R,\delta}(x)|_{\mathscr{L}(V)}\leqslant|D^2_{xx} V_R(x)|_{\mathscr{L}(V)}$ for $x\in V\backslash\{0\}$ and $D_x V_{R,\delta}$ and $D^2_{xx} V_{R,\delta}$ are bounded on V.

Apply an Itô formula for fractional Brownian motion ([6]) to the function $(V_{R,\delta}(X(t))\mathrm{e}^{2p\int_0^s\omega(t)\,\mathrm{d}s}, t\geqslant 0)$ and take the expectation to obtain

$$\begin{aligned}
&\mathbb{E}\left[V_{R,\delta}(X(t))\exp\left(2p\int_0^t\omega(s)\,\mathrm{d}s\right)\right]\\
&\quad= V_{R,\delta}(x_0)+\mathbb{E}\left[\int_0^t 2p\mathrm{e}^{2p\int_0^s\omega}\omega(s)V_{R,\delta}(X(s))+\mathrm{e}^{2p\int_0^s\omega}\left[\langle A(s)X(s), D_x V_{R,\delta}(X(s))\rangle\right.\right.\\
&\qquad\left.\left.+\langle D^2_{xx}V_{R,\delta}(X(s))BX(s), D^\phi_s(X(s))\rangle\right]\mathrm{d}s\right],
\end{aligned}\tag{3.15}$$

where

$$\begin{aligned}
D^\phi_s(X(s)) &:= \int_0^t D_q X(s)\phi_H(s-q)\,\mathrm{d}q\\
&= \int_0^t D_q\big(S(\beta^H(s))U(s,0)x_0\big)\phi_H(s-q)\,\mathrm{d}q\\
&= \int_0^t S(\beta^H(s))U(s,0)x_0 1_{[0,s]}(q)\phi_H(s-q)\,\mathrm{d}q\\
&= Hs^{2H-1}X(s)
\end{aligned}\tag{3.16}$$

for $s\in[0,t]$.

If $|X(s)|\in(\delta,R)$, then the integrand on the right hand side of (3.15) is

$$\begin{aligned}
&2p\omega(s)|X(s)|^{2p}\mathrm{e}^{2p\int_0^s\omega}\\
&\quad+\mathrm{e}^{2p\int_0^s\omega}[2\langle A(s)X(s),X(s)\rangle|X(s)|^{2(p-1)}\\
&\quad+4p(p-1)|X(s)|^{2(p-2)}\langle X(s),BX(s)\rangle\langle X(s),BX(s)\rangle s^{2H-1}H\\
&\quad+2p|X(s)|^{2p(p-1)}\langle BX(s),X(s)\rangle Hs^{2H-1}]\\
&\leqslant 2p\omega(s)|X(s)|^{2p}\mathrm{e}^{2p\int_0^s\omega}\\
&\quad+2p\mathrm{e}^{2p\int_0^s\omega}|X(s)|^{2(p-2)}[\langle A(s)X(s),X(s)\rangle|X(s)|^2\\
&\quad+Hs^{2H-1}2(p-1)\langle X(s),BX(s)\rangle^2\\
&\quad+Hs^{2H-1}|X(s)|^2|BX(s)|^2]\\
&\leqslant \mathrm{e}^{2p\int_0^t\omega}[2p\omega(s)|X(s)|^{2p}+2p|X(s)|^{2(p-2)}(-\omega(s))|X(s)|^4]\\
&\leqslant 0.
\end{aligned}\tag{3.17}$$

For a fixed $x_0 \in D$, $|x_0| \in (\delta, R)$, let

$$\Omega_R = \left\{ \sup_{s \in [0,t]} |X(s)| > R \right\}$$

and

$$\Omega_\delta = \left\{ \inf_{s \in [0,t]} |X(s)| < \delta \right\}.$$

By the sample path continuity of $(X(s), s \geqslant 0)$ and the nonattainability of $\{0\}$ the following equalities are satisfied:

$$\lim_{R \to \infty} \mathbb{P}(\Omega_R) = 0 \tag{3.18}$$

$$\lim_{\delta \downarrow 0} \mathbb{P}(\Omega_\delta) = 0. \tag{3.19}$$

From the properties of V_R and $V_{R,\delta}$ and the boundedness of $(A(s), s \geqslant 0)$ in D, the integral on the right hand side of (3.15) is bounded above by $c(|AX(s)|^{2p} + 1)$ for some constant c. This upper bound with (3.17) implies the inequality

$$\begin{aligned} \mathbb{E}\left[V_{R,\delta}(X(t)) \mathrm{e}^{2p \int_0^t \omega} \right] \leqslant V_{R,\delta}(x_0) &+ \mathbb{E}\left[1_{\Omega_R} c \int_0^t |AX(s)|^{2p} \,\mathrm{d}s \right] \\ &+ \mathbb{E}\left[1_{\Omega_\delta} c \int_0^t |AX(s)|^{2p} \,\mathrm{d}s \right]. \end{aligned} \tag{3.20}$$

For each $m > 0$, there is the inequality

$$\sup_{s \in [0,t]} \mathbb{E}|AX(s)|^m \leqslant \sup_{s \in [0,t]} [\mathbb{E}|S(\beta^H(s))|^m_{\mathscr{L}(V)} \times |U(s,0)|^m_{\mathscr{L}(V)} |x_0|^m_D] < \infty. \tag{3.21}$$

By passage to the limit as $\delta \downarrow 0$, using the Dominated Convergence Theorem on the left hand side and the Cauchy–Schwarz inequality with (3.18) and (3.19) on the right hand side, it follows that

$$\mathbb{E} V_R(X(t)) \mathrm{e}^{2p \int_0^t \omega} \leqslant V_R(x_0) + c\mathbb{E} 1_{\Omega_R} \int_0^t |AX(s)|^{2p} \,\mathrm{d}s. \tag{3.22}$$

Note that if $p \geqslant 1$ then the nonattainability of $\{0\}$ is not required because V_R is smooth at zero, so inequality (3.22) is obtained by the above procedure for V_R instead of $V_{R,\delta}$ and letting $R \to \infty$ to obtain the inequality

$$\mathbb{E} V(X(t)) \mathrm{e}^{2p \int_0^t \omega} \leqslant V(x_0) \tag{3.23}$$

which is (3.12). If $x_0 \in V$, then there is a sequence of strong solutions $(X_n(t), t \geqslant 0)$ with the D-valued initial conditions $(X_n(0) = x_n, n \in \mathbb{N})$ such that $x_n \to x_0$ in V and $X_n(t) \to X(t)$ a.s. $\square$

The following result is a $2p$-stability in the mean for Eq. (2.1).

Theorem 3.3. *If* (H1), (H2) *and* (A2) *are satisfied and there is a function* $V \in C^2(V)$ *such that* $0 \leqslant V(x) \leqslant k_1|x|^2$, $x \in V$ *for some constant* k_1,

$$|D_x V(x)| + |D^2_{xx} V(x)| \leqslant k_2(1 + |x|^m), \quad x \in V$$

for some constants k_2 *and* m *in* $(0, \infty)$ *and*

$$[\mathscr{L} V](t, x) := \langle A(t)x, D_x V(x)\rangle + Ht^{2H-1}\langle D^2_{xx} V(x)Bx, Bx\rangle \leqslant -\alpha|x|^2 \tag{3.24}$$

for $x \in D$ *and some* $\alpha > 0$, *then* Eq. (2.1) *is 2-stable in the mean. In particular*,

$$\mathbb{E}\int_0^\infty |X(t)|^2 \,\mathrm{d}t \leqslant \frac{V(x_0)}{\alpha} \leqslant \frac{k_1}{\alpha}|x_0|^2 \tag{3.25}$$

for $x_0 \in V$.

Proof. The proof is similar to the proof of Theorem 3.2, so it is only sketched. Initially the function V is replaced by V_R that differs from V only outside of $\{|x| \leqslant R\}$ to which an Itô formula with the use of expectation is applicable. Then let $R \to \infty$ in the same way as in the above proof. Using inequality (3.24) it follows that

$$0 \leqslant \mathbb{E}V(X(t)) \leqslant V(x_0) - \alpha\mathbb{E}\int_0^t |X(s)|^2 \,\mathrm{d}s \tag{3.26}$$

for $t \geqslant 0$ which implies (3.25). □

Note that Theorem 3.2 could have been formulated in terms of a general Lyapunov function V satisfying some conditions and the Lyapunov inequality. However, it would be necessary to assume that $k_1|x|^{2p} \leqslant V(x) \leqslant k_2|x|^{2p}$ for $x \in V$ and some constants k_1 and k_2 in $(0, \infty)$ (as in the well known finite dimensional case) so it seems reasonable to let $V(x) = |x|^{2p}$. However, in Theorem 3.3 the lower bound on V is not necessary, only $V \geqslant 0$, which may be useful as is shown in the following corollary. It may be applicable to the case where $A(t)$ and B are differential operators and the order of B is at most one half of the order of $A(t)$.

Corollary 3.4. *If* (H1), (H2) *and* (A2) *are satisfied and Eq.* (2.1) *has the form*

$$\begin{aligned} &\mathrm{d}X(t) = -a(t)AX(t)\,\mathrm{d}t + BX(t)\,\mathrm{d}\beta^H(t), \\ &X(0) = x_0, \end{aligned} \tag{3.27}$$

for $t \geqslant 0$ *where* $A = -A(0)$ *as above*, $A = A^*$ *and* $a : \mathbb{R}_+ \to \mathbb{R}_+$ *is strictly positive and Hölder continuous and it is assumed that* $B \in \mathscr{L}(V, \mathrm{Dom}(A^{-1/2}))$ *and*

$$-a(t) + Ht^{2H-1}|B|^2_{\mathscr{L}(V,\mathrm{Dom}(A^{-1/2}))} \leqslant -\alpha < 0 \tag{3.28}$$

for $t \in \mathbb{R}_+$ *for some* $\alpha > 0$, *then Eq.* (2.1) *are 2-stable in the mean.*

Proof. Theorem 3.2 can be applied with $V(x) = (1/2)\langle A^{-1}x, x\rangle$ for $x \in V$. Clearly $D_x V(x) = A^{-1}x$ and $D^2_{xx}V(x) = A^{-1}$ so

$$\begin{aligned}[\mathscr{L}V](t,x) &= -a(t)|x|^2 + Ht^{2H-1}\langle A^{-1}Bx, Bx\rangle \\ &\leqslant -a(t)|x|^2 + Ht^{2H-1}|B|^2_{\mathscr{L}(V,\mathrm{Dom}(A^{-1/2}))}|x|^2 \\ &\leqslant -\alpha|x|^2 \end{aligned} \tag{3.29}$$

for $x \in D$ and (3.24) is satisfied. □

Remark 3.5. Since the sample paths of the process $(X(t), t \geqslant 0)$ are (a.s.) continuous by (2.14), if this solution is pathwise exponentially stable (e.g., Proposition 3.1) with $\omega(t) \geqslant \omega_0 > 0$, then for each $\varepsilon > 0$ there is an $R > 0$ such that

$$\mathbb{P}\left(\sup_{|x_0|\leqslant 1}\ \sup_{t\geqslant 0} |X(t)| \geqslant R\right) \leqslant \varepsilon \tag{3.30}$$

that is, there is boundedness in probability. Since these solutions are linear with respect to the initial condition, for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\mathbb{P}\left(\sup_{t\geqslant 0} |X(t)| \geqslant \varepsilon\right) \leqslant \varepsilon \tag{3.31}$$

for $|x_0| \leqslant \delta$, that is, the solutions are stable with probability 1.

Some examples are considered that are motivated by the previous examples in Section 2.

Example 3.6. Consider the parabolic equation

$$\frac{\partial u(t,\xi)}{\partial t} = \sum_{i,j=1}^{d} \frac{\partial}{\partial \xi_i}\left(a_{ij}(t,\xi)\frac{\partial u(t,\xi)}{\partial \xi_j}\right) + c(t,\xi)u(t,\xi) + bu(t,\xi)\frac{\mathrm{d}\beta^H(t)}{\mathrm{d}t},$$

$$u(0,\xi) = x_0(\xi),$$

$$u|_{\mathbb{R}_+\times\partial\mathscr{O}} = 0 \tag{3.32}$$

for $(t,\xi) \in \mathbb{R}_+ \times \mathscr{O}$ on a bounded domain $\mathscr{O} \subset \mathbb{R}^d$ with a smooth boundary. It is a special case of the equation considered in Example 2.4 so the coefficients $(a_{ij}\ i,j \in \{1,\ldots,d\})$ and c are assumed to satisfy the smoothness conditions given there, and

$$\sum_{i,j=1}^{d} a_{ij}(t,\xi)v_i v_j \geqslant a_0(t)|v|^2 \tag{3.33}$$

for $v \in \mathbb{R}^d$ and $(t,\xi) \in \mathbb{R}_+ \times \mathscr{O}$ where $a_0(t) > 0$ for $t \in \mathbb{R}_+$, $a_0(\cdot)$ is continuous and

$$c(t,\xi) \leqslant c_0(t) \tag{3.34}$$

for $(t,\xi)\in\mathbb{R}_+\times\mathcal{O}$ and $c_0\in C(\mathbb{R})$. For each $x\in D=H^2(\mathcal{O})\cap H_0^1(\mathcal{O})$ it follows that

$$\begin{aligned}\langle A(t)x,x\rangle &= \int_{\mathcal{O}} x(\xi)\left(\sum_{i,j=1}^{d}\frac{\partial}{\partial\xi_i}\left(a_{ij}(t,\xi)\frac{\partial x(\xi)}{\partial\xi_j}\right)+c(t,\xi)x(\xi)\right)\mathrm{d}\xi\\ &= \int_{\mathcal{O}}\left(-\sum_{i,j=1} a_{ij}(t,\xi)\frac{\partial x(\xi)}{\partial\xi_i}\frac{\partial x(\xi)}{\partial\xi_j}+c(t,\xi)x(\xi)\right)\mathrm{d}\xi\\ &\leqslant -a_0(t)\int_{\mathcal{O}}|\nabla x(\xi)|^2 d\xi + c_0(t)|x|^2\\ &\leqslant(-a_0(t)\alpha_0+c_0(t))|x|^2, \end{aligned}\tag{3.35}$$

where $\alpha_0>0$ is the first eigenvalue of the Dirichlet Laplacian on the domain $\mathcal{O}$. This approach can be used to verify the Lyapunov inequalities in Proposition 3.1 to obtain the pathwise exponential stability and in Theorem 3.2 to obtain the moment stability, for example, if the coefficients a_{ij} and c are independent of $t\in\mathbb{R}_+$ then Eq. (2.1) is always pathwise exponentially stable and (cf. Remark 3.3(ii)) it is always stable in the $2p$th moment for $p<1/2$ if $b\neq 0$.

The pathwise stabilizing effect of this fractional Gaussian noise is more general.

Remark 3.7. Consider the equation

$$\mathrm{d}X(t)=A_0X(t)\,\mathrm{d}t+bX(t)\,\mathrm{d}\beta^H(t),$$

$$X(0)=x,$$

where A_0 is the generator of a strongly continuous semigroup S_{A_0} and $b\in\mathbb{R}\backslash\{0\}$. Solution (2.14) is pathwise exponentially stable. The solution (2.14) is given by

$$\begin{aligned}X(t) &= \exp[b\beta^H(t)]U_0(t,0)x_0\\ &= \exp\left[b\beta^H(t)-\frac{1}{2}b^2t^{2H}\right]S_{A_0}(t)x_0.\end{aligned}\tag{3.36}$$

Since $|S_{A_0}(t)|_{\mathscr{L}(V)}\leqslant K\mathrm{e}^{\omega t}$ for some $K\geqslant 1$ and $\omega\in\mathbb{R}$, the pathwise exponential stability follows from the Law of the Iterated Logarithm for a fractional Brownian motion [12].

Example 3.8. Consider the equation in Example 2.5 with the uniform ellipticity condition (2.40), that is, the parabolic Cauchy problem

$$\frac{\partial u}{\partial t}=\sum_{i,j=1}^{d} a_{ij}(t,\xi)\frac{\partial^2 u}{\partial\xi_i\partial\xi_j}+\sum_{i=1}^{d} d_i(t)\frac{\partial u}{\partial\xi_i}+c(t)u+\sum_{i=1}^{d} b_i\frac{\partial u}{\partial\xi_i}\frac{\mathrm{d}\beta_i^H(t)}{\mathrm{d}t}\tag{3.37}$$

for $(t,\xi)\in\mathbb{R}_+\times\mathbb{R}^d$ and $u(0,\xi)=x_0(\xi)$ where the coefficients are Hölder continuous on $\mathbb{R}_+$. By the conclusion of Example 1.5, there is a solution to (3.37) on $\mathbb{R}_+$. The solution can be expressed as

$$X(t)=S_1(t)U(t,0)x_0,\tag{3.38}$$

where S_1 is the shift operator defined in (2.37) which does not change the $V = L^2(\mathbb{R}^d)$ norm and $(U(t,0), t \geqslant 0)$ is the fundamental solution to the equation

$$\frac{\partial u}{\partial t} = \sum_{i,j=1}^{d} \tilde{a}_{ij}(t) \frac{\partial^2 u}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{d} d_i(t) \frac{\partial u}{\partial \xi_i} + c(t, \xi) u \tag{3.39}$$

on $\mathbb{R}^d$ where $\tilde{a}_{ij}(t) = a_{ij}(t) - Ht^{2H-1} b_i b_j$ for $i,j \in \{1, \dots, d\}$. Thus the stability analysis is reduced to the deterministic problem.

Example 3.9. In Remark 3.7 it is noted that if $A(t) = A_0$ does not depend on $t \in \mathbb{R}_+$, then the fractional noise which is defined by multiplication by a scalar b has a stabilizing effect.

If a coupled system of such equations is considered then this stabilizing property may no longer be true. Consider the following simple example:

$$\frac{\partial u_1}{\partial t} = \Delta u_1 + b_1 u_2 \frac{\mathrm{d}\beta^H}{\mathrm{d}t},$$

$$\frac{\partial u_2}{\partial t} = \Delta u_2 + b_2 u_1 \frac{\mathrm{d}\beta^H}{\mathrm{d}t}, \tag{3.40}$$

on $\mathbb{R}_+ \times \mathcal{O}$ where $\mathcal{O}$ is a regular bounded domain in $\mathbb{R}^d$, Δ is the Dirichlet Laplacian and $b_1, b_2 \in \mathbb{R} \backslash \{0\}$ satisfy $b_1 b_2 < 0$. In the deterministic case $(b_1 = b_2 = 0)$ the solution is clearly exponentially stable. For the stochastic case, $V = (L^2(\mathcal{O}))^2$,

$$B = \begin{pmatrix} 0 & b_1 I \\ b_2 I & 0 \end{pmatrix}$$

and $B^2 = b_1 b_2 I$. So B^2 is a negative operator and it is shown that it has a destabilizing effect. Solution (2.14) can be expressed as

$$X(t) = \exp\left[-\frac{1}{2} b_1 b_2 t^{2H}\right] S_B(\beta_t^H) S_A(t) x_0,$$

where S_B is the group generated by B which is periodic in time and S_A is the semigroup generated by

$$A_0 = \begin{pmatrix} \Delta & 0 \\ 0 & \Delta \end{pmatrix}.$$

For example, let x_0 be an eigenvector of A_0 corresponding to the first eigenvalue $\alpha_0 > 0$ and for notational simplicity let $b_1 = 1$ and $b_2 = -1$ so that the solution is

$$X(t) = \exp\left[\frac{1}{2} t^{2H} - \alpha_0 t\right] S_B(\beta^H(t)) x_0,$$

where

$$S_B(s) = \sqrt{2}\begin{pmatrix} \sin\left(s+\frac{\pi}{4}\right) & 0 \\ 0 & \cos\left(s+\frac{\pi}{4}\right) \end{pmatrix} I.$$

If $x_0 = (x_0^1, x_0^2)$ where $x_0^1 = x_0^2 \in L^2(\mathcal{O})\backslash\{0\}$, it is clear that $|S_B(s)x_0|^2 = |x_0|^2$ for each $s \in \mathbb{R}$ so that

$$\lim_{t\to\infty} |X(t)| = \infty \quad \text{a.s.}$$

A similar destabilizing effect may occur for a single equation as is described in the following example of a parabolic equation of fourth order.

Example 3.10. Consider the Cauchy problem

$$\frac{\partial u(t,\xi)}{\partial t} = -\frac{\partial^4 u}{\partial \xi^4} - \alpha u + \frac{\partial u}{\partial \xi}\frac{\mathrm{d}\beta^H(t)}{\mathrm{d}t},$$

$$u(0,\xi) = x_0(\xi), \tag{3.41}$$

for $(t,\xi) \in \mathbb{R}_+ \times \mathbb{R}$ in the weighted space $V = L^2_\rho(\mathbb{R})$ with the weight $\rho(\xi) = \mathrm{e}^{-K|\xi|}$ for $\xi \in \mathbb{R}$ and a fixed $K>0$. It is well known (e.g., [22]) that the operator $L(u) = -(\partial^4 u/\partial\xi^4) - \alpha u$ generates a C_0-semigroup in the space V which is exponentially stable for $\alpha>0$. If is shown that adding a fractional noise in (3.41) may destabilize the solution. As noted in the above examples, $B = \partial/\partial\xi$ generates a C_0-group S_1 in V which is a shift operator

$$[S_1(t)x](\xi) = x(t+\xi)$$

for $t,\xi \in \mathbb{R}$. This group commutes with L. The operator $\tilde{A}(t)$ is

$$\tilde{A}(t) = L - tH^{2H-1}B^2 = -\frac{\partial^4}{\partial\xi^4} - \alpha I - tH^{2H-1}\frac{\partial^2}{\partial\xi^2} \tag{3.42}$$

so $(\tilde{A}(t), t\geqslant 0)$ is strongly elliptic and generates a strongly continuous evolution system $(U(t,s), 0\leqslant s\leqslant t)$. Conditions (A1)–(A3) are satisfied and there is a solution (2.14) to Eq. (3.41) which can be expressed as

$$X(t) = S_1(\beta^H(t))U(t,0)x_0. \tag{3.43}$$

Choose the initial condition $x_0(\xi) = \sin\xi$ and let $y(t,\xi) = [U(t,0)x_0](\xi)$. Expressing y as $y(t,\xi) = \varphi(t)\sin\xi$ compute the function φ that satisfies the differential equation

$$\dot\varphi(t)\sin\xi = -\varphi(t)\sin\xi - \alpha\varphi(t)\sin\xi + Ht^{2H-1}\varphi(t)\sin\xi,$$

$$\varphi(0) = 1.$$

Thus $\varphi(t) = \exp[-t-\alpha t + (1/2)t^{2H}]$. By (2.14)

$$X(t) = \sin(\xi + \beta^H(t))\exp\left[-(1+\alpha)t + \frac{1}{2}t^{2H}\right]. \tag{3.44}$$

Since

$$\inf_{s\in\mathbb{R}} |\sin(\cdot+s)|^2 = \inf_{s\in[0,2\pi]} \int_{\mathbb{R}} \mathrm{e}^{-K|\xi|}\sin^2(\xi+s)\,\mathrm{d}\xi > 0$$

so it follows directly that

$$\lim_{t\to\infty}|X(t)| = \infty \quad \text{a.s.}$$

However, this does not occur for all initial conditions. For example, if $x_0 \equiv 1$ then $y(t,\xi) = \mathrm{e}^{-\alpha t}$ and $\lim_{t\to\infty} X(t) = 0$ a.s.

Example 3.11. This example considers the problem of identification of some unknown parameters for a stochastic parabolic equation with a scalar fractional Brownian motion. Consider Eq. (3.32) where, as in Examples 3.6 and 2.4, the coefficients are assumed to be sufficiently smooth and the ellipticity condition (3.33) is satisfied. The problem is to estimate the unknown constant b in the diffusion term from the observations of the solution in the time interval $[0,T]$. Let $A(t)$ be the differential operator

$$A(t)u = \sum_{i,j} \frac{\partial}{\partial \xi_i}\left(a_{ij}(t,\xi)\frac{\partial u}{\partial \xi_j}\right) + c(t,\xi)u, \tag{3.45}$$

where $\mathrm{Dom}(A(t)) = H^2(\mathcal{O}) \cap H_0^1(\mathcal{O})$. The family $(A(t), t\geqslant 0)$ generates the strong evolution system $(U_0(t,0), 0\leqslant s\leqslant t)$ where $V = L^2(\mathcal{O})$. Clearly solution (2.14) can be express as

$$X(t) = u(t,\xi) = \exp\left[b\beta^H(t) - \frac{1}{2}b^2t^{2H}\right][U_0(t,0)x_0](\xi), \tag{3.46}$$

for $\xi \in \mathcal{O}$. Fix $\xi_0 \in \mathcal{O}$ and assume that $x_0 \in C(\bar{\mathcal{O}})$, $x_0(\xi)\geqslant 0$, $\xi \in \mathcal{O}$ and $x_0 \not\equiv 0$. By the strong maximum principle, there is the inequality

$$[U_0(t,0)]x(\xi_0) > 0$$

for $t>0$ and by (3.46) $u(t,\xi_0)>0$ a.s. Let $(P_n, n\in\mathbb{N})$ be a sequence of nested partitions of $[0,T]$ that becomes arbitrarily fine in $[0,T]$ and $P_n = \{t_0^{(n)},\dots,t_n^{(n)}\}$. Since the function $t\mapsto U_0(t,0)x(\xi_0)$ is differentiable and

$$\log u(t,\xi_0) = b\beta^H(t) - \tfrac{1}{2}b^2t^{2H} + \log([U_0(t,0)x_0](\xi_0)) \tag{3.47}$$

it follows that

$$\lim_{n\to\infty}\sum_{i=1}^{n-1} |\log(u(t_{i+1}^{(n)},\xi_0)) - \log(u(t_i^{(n)},\xi_0))|^{1/H} = |b|^{1/H}\mathbb{E}|\beta^H(T)|^{1/H} \quad \text{a.s.}$$

and thus $|b|$ is identified.

Now assume that the matrix (a_{ij}) is constant, $c = 0$ and the equation is

$$\mathrm{d}X(t) = aAX(t)\,\mathrm{d}t + bX(t)\,\mathrm{d}\beta^H(t),$$

$$X(0) = x_0, \tag{3.48}$$

where $A = A(t)$ is given by (3.45) does not depend on time and $a > 0$ is an unknown parameter. The operator A is self-adjoint, negative and A^{-1} is compact so there is a countable orthonormal basis $(e_n, n \in \mathbb{N})$ of eigenvectors of A and the corresponding (positive) eigenvalues $(\alpha_n, n \in \mathbb{N})$ that diverge such that

$$Ae_n = -\alpha_n e_n$$

for $n \in \mathbb{N}$. For $x_0 = e_n$ for some fixed $n \in \mathbb{N}$, (3.46) implies that

$$X(t) = \exp\left[b\beta^H(t) - \frac{1}{2}b^2 t^{2H} - a\alpha_n t\right] e_n. \tag{3.49}$$

If $x_0 \neq 0$ then there is an $m \in \mathbb{N}$ such that $\langle e_m, x_0\rangle \neq 0$. By (3.49) it follows that

$$\langle X(t), e_m\rangle = \exp\left[b\beta^H(t) - \frac{1}{2}b^2 t^{2H} - a\alpha_m t\right]\langle e_m, x_0\rangle \tag{3.50}$$

so $\langle X(t), e_m\rangle \neq 0$ and the pair $\langle X(t), e_m\rangle$ and $\langle x_0, e_m\rangle$ have the same sign. By (3.50) there is the equality

$$\log\left(\frac{\langle X(t), e_m\rangle}{\langle x_0, e_m\rangle}\right) = b\beta^H(t) - \frac{1}{2}b^2 t^{2H} - a\alpha_m t$$

for $t \geqslant 0$. If a family of estimators $(\hat{a}(t), t \geqslant 0)$ is defined as

$$\hat{a}(t) := \frac{1}{-\alpha_m t}\left(\log\left(\frac{\langle X(t), e_m\rangle}{\langle x_0, e_m\rangle}\right) + \frac{1}{2}b^2 t^{2H}\right). \tag{3.51}$$

then the Law of the Iterated Logarithm for β^H implies the strong consistency of $(\hat{a}(t), t \geqslant 0)$, that is,

$$\lim_{t\to\infty} \hat{a}(t) = a \quad \text{a.s.}$$

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